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K-Rotund Complex Normed Linear Spaces

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In this paper, we consider K-rotundity in a complex normed linear space and prove a theorem which is the generalization of the Taylor–Foguel theorem. We also consider the relations between K-uniform rotundity, fully K-convexity, and K-rotundity. © 1990 Academic Press, Inc.

P. R. Beesack, E. Huges, and M. Ortel have already shown [1] that rotundity in a complex normed linear space is equivalent to the property that for any distinct vectors x and y of unit norm, a complex number α may be found for which $\|\alpha x+(1-\alpha) y\|<1$. This leads to a new proof of a result due to Taylor and Foguel on the uniqueness of Hahn–Banach extensions.

In this paper, we introduce a notion of K-rotundity in a complex normed linear space and prove a theorem that generalizes the theorem of Taylor and Foguel on the uniqueness of Hahn–Banach extensions.

Without special explanation, X is a normed linear space over the complex field $\mathbb{C}$, X^* is the conjugate space of X. $S(X)=\{x\in X\colon \|x\|=1\}$, $B(X)=\{x\in X\colon \|x\|\leqslant 1\}$.

If $f\in X^*$, then $N(f)=\{x\in X\colon f(x)=0\}$ is the null space of f, and it is a subspace of X with co-dimension equal to 1. It is well known that for any pair of elements $f, g\in X^*$, they are linearly dependent if and only if $N(f)=N(g)$.

If A is any subset of X, then span A denotes the set of all linear combinations of A. Fix any $x\in X-N(f)$, it is clear that $X=\operatorname{span}\{x, N(f)\}$.

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DEFINITION 1. Let K be a positive integer, and $\dim X \geqslant K+1$. A point $x \in S(X)$ is called a K-extreme point of $B(X)$ iff x cannot be represented as $(1/(K+1))(x_1 + x_2 + \cdots + x_{K+1})$, where $x_i \in S(X)$, $i = 1, 2, \ldots, K+1$, and the set $x_1, x_2, \ldots, x_{K+1}$ is linearly independent.

DEFINITION 2. Let K be a positive integer, and $\dim X \geqslant K+1$. X is said to be K-rotund iff each point of $S(X)$ is a K-extreme point of $B(X)$.

Obviously, a 1-rotund complex normed linear space is just a rotund complex normed linear space.

THEOREM 1. *Let K be a positive integer, and* $\dim X \geqslant K+1$. *Then X is K-rotund if and only if X has the property (L_K): Whenever $x_i \in S(X)$, $i = 1$, $2, \ldots, K+1$, forming a linearly independent set, we have $\|\sum_{i=1}^{K+1} \alpha_i x_i\| < 1$ for some $\alpha_i \in \mathbb{C}$, $i = 1, 2, \ldots, K+1$, with $\alpha_1 + \alpha_2 + \cdots + \alpha_{K+1} = 1$.*

Proof. Evidently, if X is K-rotund, then the property (L_K) holds with $\alpha_i = 1/(K+1)$, $i = 1, 2, \ldots, K+1$.

Now we prove that if X is not K-rotund then X does not have property (L_K). Suppose there is a linearly independent set $x_i \in S(X)$, $i = 1, 2, \ldots, K+1$ and $\|(x_1 + x_2 + \cdots + x_{K+1})/(K+1)\| = 1$. By the Hahn–Banach theorem, there is an $f \in X^*$ such that $\|f\| = 1 = f((x_1 + x_2 + \cdots + x_{K+1})/(K+1))$, i.e., $(1/(K+1))/[f(x_1) + f(x_2) + \cdots + f(x_{K+1})] = 1$. Since $|f(x_i)| \leqslant 1$, $i = 1, 2, \ldots, K+1$, by the proposition which is easily shown that "if $a_1 + a_2 + \cdots + a_K = K$, $a_i \in \mathbb{C}$, $|a_i| \leqslant 1$, $i = 1, 2, \ldots, K$, then $a_1 = a_2 = \cdots = a_K = 1$," it follows that $f(x_1) = f(x_2) = \cdots = f(x_{K+1}) = 1$. Hence for any complex numbers α_i, $i = 1, 2, \ldots, K+1$, with $\alpha_1 + \alpha_2 + \cdots + \alpha_{K+1} = 1$, we have $\|\sum_{i=1}^{K+1} \alpha_i x_i\| \geqslant |f(\sum_{i=1}^{K+1} \alpha_i x_i)| = 1$, and so property (L_K) fails.

This completes the proof.

THEOREM 2. *If for some positive integer K, X is K-rotund, then X is $(K+1)$-rotund.*

Proof. Suppose that X is not $(K+1)$-rotund. Then there exists a point $x_0 \in S(X)$ and a linearly independent set $x_i \in S(X)$, $i = 1, 2, \ldots, K+2$, such that $x_0 = (1/(K+2))(x_1 + \cdots + x_{K+2})$. Let $y_i = (1/(K+2))[(K+1) x_i + x_{K+2}]$, $i = 1, 2, \ldots, K+1$. By Theorem 1, $y_i \in S(X)$, $i = 1, 2, \ldots, K+1$. Clearly, $y_1, y_2, \ldots, y_{K+1}$ form a linearly independent set, and $x_0 = (1/(K+1))(y_1 + y_2 + \cdots + y_{K+1})$. Hence X is not K-rotund. The theorem is proved.

As an application of Theorem 1 we prove the following result which is the generalization of a theorem due to Taylor [2] and Foguel [3].

THEOREM 3. *Let K be a positive integer, and* $\dim X \geqslant K+1$. *Then all bounded linear functionals defined on subspaces of X have at most K linearly independent norm-preserving linear extensions to X if, and only if, the conjugate space X^* is K-rotund.*

Proof. Suppose X^* is K-rotund but there are a subspace M of X and a bounded linear functional g on M having $f_i \in X^*$, $i = 1, 2, ..., K+1$, as $K+1$ linearly independent norm-preserving linear extensions to X. We may assume that $\|g\| = 1$. Clearly, now $f = \sum_{i=1}^{K+1} \alpha_i f_i$ is a linear extension of g for any $\alpha_i \in \mathbb{C}$ with $\alpha_1 + \cdots + \alpha_{K+1} = 1$. By K-rotundity we have

$$\left\| \sum_{i=1}^{K+1} \alpha_i f_i \right\| < 1$$

for some $\alpha_i \in \mathbb{C}$ with $\alpha_1 + \alpha_2 + \cdots + \alpha_{K+1} = 1$. This is impossible since, for any $\alpha_i \in \mathbb{C}$ with $\alpha_1 + \cdots + \alpha_{K+1} = 1$, we have

$$\left\| \sum_{i=1}^{K+1} \alpha_i f_i \right\| \geqslant \sup\{|g(x)| : x \in M, \|x\| = 1\} = 1.$$

Conversely, let f_i $(1 \leqslant i \leqslant K+1)$ be independent elements of $S(X^*)$. Clearly $f_i - f_{K+1}$ $(1 \leqslant i \leqslant K)$ are independent too. Find x_i with $(f_i - f_{K+1})(x_j) = \delta_{ij}$ (see Kelley, "General Topology," p. 108, problem $\omega_{(a)}$). If $M = \bigcap_{i=1}^{K} N(f_i - f_{i+1})$, then $x = \sum_{i=1}^{K} a_i x_i + m$, $m = m_x \in M$, $a_i = (f_i - f_{K+1})(x)$, $\forall x \in X$. Let g be the restriction of f_i to M and h a norm-preserving extension of g. Note $\|g\| < 1$: if we had $\|g\| = 1$ then f_i $(1 \leqslant i \leqslant K+1)$ are $K+1$ norm-preserving independent extensions. Then $\alpha_i \in \mathbb{C}$, $\sum_{i=1}^{K+1} \alpha_i = 1$, $h = \sum_{i=1}^{K+1} \alpha_i f_i$ with $\alpha_i = h(x_i) - f_{K+1}(x_i)$ $(1 \leqslant i \leqslant K)$ and $\alpha_{K+1} = 1 - \sum_{i=1}^{K} \alpha_i$. Since $\|h\| = \|\sum_{i=1}^{K+1} \alpha_i f_i\| < 1$, then, by Theorem 1, X^* is K rotund.

The proof is now complete.

COROLLARY 4. *Let* $\dim X \geqslant 2$. *Then all bounded linear functionals defined on subspaces of X have unique norm-preserving linear extensions to X if, and only if, the conjugate space X^* is rotund.*

Now let X be a real Banach space, and let us consider the relations between K-uniform rotundity, fully K-convexity, and K-rotundity in X.

DEFINITION 3 [4]. Let $K \geqslant 1$. A Banach space X is called K-locally uniformly rotund $(K-LUR)$ if for any $x \in S(X)$ and $\varepsilon > 0$, there exists

$\delta>0$ such that for any $x_i \in S(X)$, $i=1,2,\ldots,K$, with $\|(1/(K+1))(x+x_1+\cdots+x_K)\| \geqslant 1-\delta$ then

$$V(x, x_1, \ldots, x_K)=\sup\left\{\left|\begin{matrix} 1 & 1 & & 1 \\ f_1(x) & f_1(x_1) & \cdots & f_1(x_K) \\ \vdots & & & \vdots \\ f_K(x) & f_K(x_1) & \cdots & f_K(x_K)\end{matrix}\right| : \begin{matrix} f_i \in X^*, \\ \|f_i\| \leqslant 1, \\ i=1,2,\ldots,K\end{matrix}\right\}<\varepsilon.$$

THEOREM 5. *If X is $K-LUR$, then X is K-rotund.*

Proof. If X is not K-rotund, then there exists $x, x_1, \ldots, x_K \in S(X)$ such that for any $\delta>0$,

$$\left\|\frac{1}{K+1}(x+x_1+\cdots+x_K)\right\|=1 \geqslant 1-\delta$$

and $x, x_1, \ldots, x_K$ are linearly independent. Hence there exists $f_1, \ldots, f_K \in S(X^*)$ such that $f_i(x)=0$, $f_i(x_j)=\delta_{ij}$, $i, j=1,2,\ldots,K$. It follows that for $\varepsilon=\frac{1}{2}$,

$$V(x, x_1, \ldots, x_K) \geqslant \left|\begin{matrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 0 & & 0 \\ 0 & 0 & 1 & & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \cdots & 1\end{matrix}\right|=1>\varepsilon.$$

Hence X is not $K-LUR$.

DEFINITION 4 [4]. Let $K \geqslant 1$ be an integer. A Banach space X is said to be K-uniformly rotund ($K-UR$) if for any $\varepsilon>0$, there exists $\delta>0$ such that for any $x_i \in S(X)$, $i=1,2,\ldots,K+1$, with $\|(1/(K+1))(x_1+\cdots+x_{K+1})\| \geqslant 1-\delta$, then $V(x_1, x_2, \ldots, x_{K+1})<\varepsilon$.

COROLLARY 6. *If X is $K-UR$, then X is K-rotund.*

DEFINITION 5 [5]. Let $K \geqslant 2$ be an integer. A Banach space X is called fully K-convex, if every sequence $\{x_n\}$ of elements in X satisfying $\lim_{n\to\infty}\|x_n\|=1$ and $\|(1/K)(x_{n_1}+\cdots+x_{n_K})\| \geqslant 1$ for any K indices $n_1 \leqslant n_2 \leqslant \cdots \leqslant n_K$ is a Cauchy sequence.

In [6], it is proved that $\forall K \geqslant 2$,

(a) $R+K-UR \Rightarrow$ fully $(K+1)$-convex,

(b) $R+K-UR \nRightarrow$ fully K-convex,

where R means "strictly convex."

From these, we obtain the following results:

(1) K-rotund $\not\Rightarrow K-UR$.

In fact, if K-rotund $\Rightarrow K-UR$, then we will have $R \Rightarrow$ fully $(K+1)$-convex because $R \Rightarrow K$-rotund and (a), it is in contradiction with (b).

(2) K-rotund $\not\Rightarrow$ fully K-convex.

In fact, if K-rotund $\Rightarrow$ fully K-convex, then $K-UR \Rightarrow$ fully K-convex by Corollary 6, and it is contradicting (b).

(3) K-rotund $\not\Rightarrow$ fully $(K+n)$-convex $(n \geqslant 1)$.

Otherwise, it follows that $R \Rightarrow$ fully $(K+n)$-convex for all K, in particular, $R \Rightarrow$ fully 2-convex, and this is a contradiction.

(4) Fully 2-convex $\not\Rightarrow$ 2-rotund.

Let $E = (l_2, \|\cdot\|)$, where

$$\|x\|^2 = [|a_1| + (a_2^2 + a_3^2 + \cdots)^{1/2}]^2 + \left[\left(\frac{a_2}{2}\right)^2 + \cdots + \left(\frac{a_n}{n}\right)^2 + \cdots\right]^2$$

for all $x = (a_1, a_2, ...)$ in E. In [7], it is proved that E is a fully 2-convex space. Now we will prove that E is not 2-rotund.

Let $x_1 = (1, 0, ...)$, $x_2 = (0, a, 0, ...)$, $x_3 = (0, b, c, 0, ...)$, where $a, b, c \neq 0$, then x_1, x_2, x_3 are linearly independent. Take a such that $\|x_2\| = 1$, i.e., $a^2 = 4\sqrt{5} - 8$. We can prove that there exist b, c which satisfy

$$\|x_3\| = 1 \qquad \text{and} \qquad \|x_1 + x_2 + x_3\| = 3,$$

i.e.,

$$b^2 + c^2 + \left(\frac{b^2}{4} + \frac{c^2}{9}\right)^2 = 1 \tag{1}$$

$$[1 + \sqrt{(a+b)^2 + c^2}]^2 + \left[\left(\frac{a+b}{2}\right)^2 + \frac{c^2}{9}\right]^2 = 9. \tag{2}$$

From (1), (2), and $a^2 + (a^2/4)^2 = 1$, we obtain

$$\sqrt{(a+b)^2 + c^2} + ab + \frac{a^3 b}{8} + \frac{3a^2 b^2}{16} + \frac{ab^3}{8} + \frac{a^2 c^2}{36} + \frac{abc^2}{18} = 3,$$

and $-(9/4)b^2 - (81/2) + (9/2)\sqrt{5b^2 + 85} = c^2$.

So $|b| < \sqrt{\sqrt{80} - 8} \approx 0.97$ because $c^2 > 0$. Let

$$f(b) = \sqrt{(a+b)^2 + c^2} + ab + \frac{a^3 b}{8} + \frac{3a^2 b^2}{16} + \frac{ab^3}{8} + \frac{a^2 c^2}{36} + \frac{abc^2}{18} - 3,$$

then

$$f(0)=\sqrt{a^2+c^2}+\frac{1}{36}a^2c^2-3<0,$$

where $c^2=(9/2)\sqrt{85}-(81/2)$,

$$f(\sqrt{0.9})>a+b+ab+\frac{ab}{8}(a^2+b^2)+\frac{3}{16}a^2b^2-3>0.$$

Hence there exists $b\in(0, \sqrt{0.9})$ such that $f(b)=0$, and it is required.

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